Discrete Comput Geom 12:91–104 (1994)

Discrete & Computational
Geometry

Camera Placement in Integer Lattices*

E. Kranakis[1] and M. Pocchiola[2]

[1] Centrum voor Wiskunde en Informatica, P.O. Box 4079,
1009 AB, Amsterdam, The Netherlands
and
School of Computer Science, Carleton University,
Ottawa, Ontario, Canada K1S 5B6
kranakis@scs.carleton.ca

[2] Département de Mathématiques et d'Informatique, Ecole Normale Supérieure,
URA 1327 CNRS, 45 rue d'Ulm, 75230 Paris Cédex 05, France
pocchiola@dmi.ens.fr

Abstract. The camera placement problem concerns the placement of a fixed number of point-cameras on the d-dimensional integer lattice in order to maximize their visibility. We reduce the problem to a finite discrete optimization problem and give a characterization of optimal configurations of size at most 3^d.

1. Introduction

Visibility and illumination problems are among the most appealing and intuitive research topics of combinatorial geometry. In many cases (though not all) their analysis requires nothing more than basic topics from geometry, number theory, and graph theory and as such they are very well suited for a wide audience [2]. In recent years there has been particular emphasis on the algorithmic component of visibility problems in polygonal configurations and as such they have come to be studied under the area of "art gallery (watchman) problems." In turn this last area lies at the intersection of combinatorial and computational geometry [12]. In this paper we focus on a particular class of art gallery problems, namely those visibility problems which concern configurations of points lying on the vertices of

* A preliminary version of this work appears in [8]. The research of E. Kranakis was supported by NSERC Grant No. 907002.

the integer lattice Λ. In particular we are interested in the following art gallery problem:

> Given an integer s, determine a configuration S (of camera locations) contained in Λ and of cardinality s, such that the density of lattice points which are visible from at least one point of S is as large as possible.

By Λ we denote the d-dimensional integer lattice consisting of d-tuples of integers and by Λ_n we denote the set of lattice points in Λ whose coordinates have absolute value $\leq n/2$. For any set $X \subseteq \Lambda$ of lattice points the density $D(X)$ of X is defined as the limit (if it exists) of the ratio $|X \cap \Lambda_n|/|\Lambda_n|$ of the number of points in $X \cap \Lambda_n$ to the number of points in Λ_n as n tends to infinity. It is easy to check that the density function is a finitely additive measure on those subsets of Λ which have density. Let $\mathscr{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers, and let p range over $\mathscr{P}$, and let $\mathscr{Q}$ range over subsets of $\mathscr{P}$. Two lattice points are called *visible modulo p* if they are distinct modulo p. Two lattice points are said to be *visible modulo* $\mathscr{Q}$ if they are visible modulo p for each prime p in $\mathscr{Q}$. Two points visible modulo $\mathscr{P}$ are visible in the geometric sense, i.e., the open line segment joining them avoids all the lattice points (see Fig. 1). For all $X \subseteq \Lambda$, X/p denotes the quotient set of X by the relation of equality modulo p.

Several visibility problems have been studied on integer lattices [3], [4]. Of these we single out two which are relevant for our study. Rumsey [15] shows that for any finite set S of lattice points, the density of the set of lattice points visible from each point of S is given by the infinite product

$$\prod_{p \in \mathscr{P}} \left(1 - \frac{|S/p|}{p^d}\right). \tag{1}$$

The above formula was previously obtained by G. Leujeune Dirichlet for the case $|S| = 1$ ("the probability that d integers chosen at random are relatively prime is $1/\zeta(d)$," where $\zeta(z) = \sum_{n \geq 1} n^{-z}$, $|z| > 1$, denotes the Riemann zeta function [7, p. 324]) and by Rearick [14] for the case where $|S| = 2$, as well as for the case where the points of S are pairwise visible. The following art gallery problem was

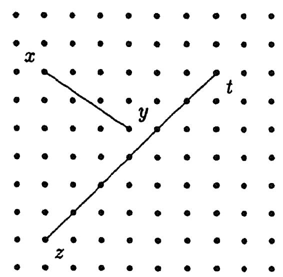

Fig. 1. Points x and y are visible; points z and t are visible modulo p for $p \neq 2, 3$.

posed by Moser [11] in 1966: Given a set P of points in the plane how many guards located at points of P are needed to see the unguarded points of P? Abbott [1] shows, in the case $P = \Lambda_n$, that the number of guards required is bounded above by $4 \ln n$ (and below by $\ln n/2 \ln \ln n$). For the upper bound Abbott constructs recursively a "greedy" sequence $x_1, x_2, \ldots, x_k$ such that, for each i, x_{i+1} is a point x in the set Λ_n for which the set-theoretic difference

$$V_n(x) \backslash (V_n(x_1) \cup \cdots \cup V_n(x_i)),$$

where $V_n(x)$ is the set of points of Λ_n visible from x, is of maximal size and shows that $k = O(\ln n)$ iterations of this procedure suffice in order to cover all the vertices of the lattice. His method however gives no "qualitative" information on the location of these points on the lattice. In some respects the camera placement problem can be thought of as a "qualitative" version of Abbott's problem. Despite the fact that Abbott's (and hence Moser's) question still remains open we expect that our investigations will also contribute to a better understanding of this problem.

The paper is organized as follows. In Section 2 we give a precise mathematical formulation of the camera placement problem. In Section 3 we show that the cameras of an optimal configuration have to be visible modulo p for each prime $p \geq s^{1/d}$; in particular, a solution to the problem exists. Then we reformulate our problem into the following integer optimization problem:

$$\text{maximize } u'(b_1) + u'(b_2) + \cdots + u'(b_m)$$
$$\text{subject to } \begin{cases} (b_1, \ldots, b_m) = B(a_1, \ldots, a_m), \\ a_1 + \cdots + a_m = s, \quad a_i \in \mathbb{N}, \end{cases} \tag{2}$$

where u' is an absolutely monotone function, B is a linear operator, and $m \in \mathbb{N}$; the three parameters u', B, and m depend on s (see Section 3 for the appropriate definitions of u', B, and m). This enables us to solve the problem, in Section 4, for the case of $s \leq 3^d$ cameras: a configuration of at most 3^d cameras is optimal if and only if its cameras are evenly distributed in the classes of Λ/p as p ranges over the set of primes; in particular, a configuration of at most 2^d cameras is optimal if and only if its cameras are pairwise visible. Figure 2 depicts an optimal configuration of 27 cameras in dimension 3.

2. Camera Placement Problem

2.1. Abstract Configurations

The *camera placement problem* in the d-dimensional integer lattice Λ is the following:

Given an integer s, determine a configuration S of s lattice points (camera

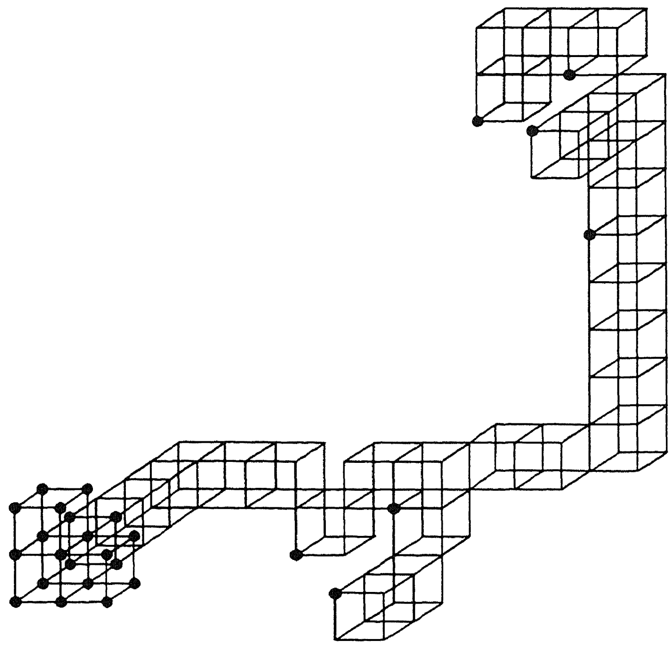

Fig. 2. An optimal configuration of 27 cameras in dimension 3.

locations) such that the density of lattice points visible by at least one point of S is maximized.

More formally, we want to find conditions on the set S of possible camera locations so that the following quantity,

$$u(S) := \sum_{E \subseteq S, E \neq \varnothing} (-1)^{|E|+1} \prod_{p \in \mathscr{P}} \left(1 - \frac{|E/p|}{p^d}\right), \tag{3}$$

which is obtained from the product formula (1) using the principle of inclusion/exclusion, is maximized. The quantity $u(S)$ is called the *visibility* of the configuration. Configurations (if any) which, for a given s, attain the optimal density are called *optimal*. Clearly, the visibility of a configuration depends only on the relations of visibility modulo p shared by its cameras as p ranges over $\mathscr{P}$. This leads to the notion of *abstract configuration.*

Definition 2.1. An abstract $\mathscr{Q}$-configuration of size s is a family of equivalence relations $(r_p)_{p \in \mathscr{Q}}$ on the set $\{1, \ldots, s\}$ indexed by the set of prime numbers in $\mathscr{Q}$ such that

$$i r_p j \quad \text{if and only if} \quad A_i - A_j \in p\Lambda,$$

for some ordered configuration $\{A_1, \ldots, A_s\}$ of s lattice points. In that case the configuration $\{A_1, \ldots, A_s\}$ is called a representative of the family $(r_p)_{p \in \mathcal{Q}}$. Two ordered configurations which represent the same abstract $\mathcal{Q}$-configuration are called equivalent modulo $\mathcal{Q}$.

In view of (3) it is obvious that two configurations which are equivalent modulo $\mathcal{P}$ have the same visibility. The camera placement problem is then split into two subproblems:

(1) Give necessary and sufficient condition for an abstract configuration to be optimal.
(2) Compute a representative of a given abstract (optimal) configuration.

In this article we focus on the first problem (in [13] it is shown that a representative of an optimal abstract configuration can be computed in expected time exponential in $s^{1/d}$ for d fixed). At this point and to guide our analysis it can be useful to make a conjecture about the solution. In view of (3) it is reasonable to believe that the cameras of an optimal configuration have to be evenly distributed in the classes of Λ/p as p ranges over the set of primes. This leads to the notion of *balanced* configuration.

Definition 2.2. A configuration S is called balanced if, for all square free integers n and for all cosets c and c' in Λ/n, we have $\||S \cap c| - |S \cap c'|\| \leq 1$.

Conjecture 2.1. *An optimal configuration is necessarily balanced.*

Before we attempt to prove (or disprove!) this conjecture we must determine which family of equivalence relations have a representative; in particular, we must determine if balanced configurations exist. We examine this question in the next subsection.

2.2. *The Realizability Theorem*

A set $X \subseteq \Lambda$ is called *periodic of period* (the natural number) m if $X = X + m\Lambda$; in other words X is the union of some classes of the quotient set of Λ by the relation of equality modulo m. It can be easily verifed that a periodic set $X \subseteq \Lambda$ of period m has a density given by the ratio $|X/m|/|\Lambda/m|$, and that the density of a finite intersection of periodic sets whose periods are pairwise relatively prime is the product of the densities of the periodic sets (see [15]). Consider now a set S of lattice points and let $V(S)$ ($V_p(S)$) be the set of lattice points which can see (modulo p) each of the points of S. Rumsey [15] observes that $V_p(S)$ is periodic of period p and that $V(S)$ is the intersection of the sets $V_p(S)$, as p ranges over $\mathcal{P}$. So it is tempting to assert (allowing the primes to "play a game of chance" [6, Chapter 4]) that the density of $V(S)$ is the product of the densities of the $V_p(S)$. The main result of [15] is to give a necessary and sufficient condition so that the above assertion is true. What is this necessary and sufficient condition? It turns out that this condition can be reformulated in a much more versatile way when we replace the sequence $V_2(S)$, $V_3(S)$, $V_5(S), \ldots$ by an arbitrary sequence X_1, X_2, $X_3, \ldots$ of periodic sets of pairwise relatively prime periods. This reformulation is the following.

Theorem 2.1. *Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of periodic sets such that X_k is periodic with respect to m_k. Assume that the m_k are pairwise relatively prime and put*

$$A(K) = \{x \in \Lambda \mid \exists k,\ x \in (X_1 \cap \cdots \cap X_{k-1} \cap \bar{X}_k),\ m_k > K|x|\}.$$

Then the following assertions are equivalent:

(i) $D(\bigcap_k X_k) = \prod_k D(X_k)$.
(ii) $\lim_{K \to \infty} \bar{D}(A(K)) = 0$.

Proof. Up to notation Rumsey's proof as given in [15] works in our present framework. We refer to [10] for the details. □

Rumsey used the above theorem to prove that the set $V(S)$ admits a density when S is finite. We reformulate this result replacing the set $\mathscr{P}$ of prime numbers by a subset $\mathscr{Q}$ of $\mathscr{P}$.

Corollary 2.1 (Rumsey's Theorem). *Let S be a finite set of lattice points and let $\mathscr{Q}$ be a set of prime numbers. The set $V_{\mathscr{Q}}(S)$ of lattice points visible modulo $\mathscr{Q}$ from each point of S admits a density given by the infinite product*

$$\prod_{p \in \mathscr{Q}} \left(1 - \frac{|S/p|}{p^d}\right).$$

Furthermore, $V_{\mathscr{Q}}(S)$ is nonempty if and only if $|S/p| < p^d$ for all $p \in \mathscr{Q}$. □

Now we come to the existence question of a representative for a family of equivalence relations.

Corollary 2.2 (Realizability Theorem). *Let $\mathscr{E} = (r_p)_{p \in \mathscr{Q}}$ be a family of equivalence relations on the set $[s] = \{1, \ldots, s\}$. A representative of $\mathscr{E}$ exists if and only if $|[s]/r_p| \leq p^d$ for all prime $p \in Q$, and $|[s]/r_p| = s$ for p large enough. Furthermore, the set of representatives of $\mathscr{E}$ admits a density given by the infinite product*

$$\prod_{p \in \mathscr{Q}} \frac{(p^d)_{|[s]/r_p|}}{p^{sd}},$$

where $(x)_y = x(x-1)\cdots(x-y+1)$ is the descent factorial.

Proof. Let X_p be the set of s-tuples $(A_1, \ldots, A_s)$ of points of Λ such that $A_i - A_j \in p\Lambda$ if and only if $i r_p j$ for all i and j. Clearly, the set, say X, of representatives of $\mathscr{E}$ is the intersection of the X_p as p ranges over $\mathscr{Q}$; furthermore, we can easily verify that X_p is periodic with period p, and that its density is given by the ratio $(p^d)_{|[s]/r_p|}/p^{sd}$. Then we are in a position to apply Theorem 2.1. First we observe that if $\prod_p D(X_p) = 0$, then our theorem is proved since $D(X) \leq \prod_p D(X_p)$. Hence without loss of generality we may assume that $\prod_p D(X_p) \neq 0$; in particular, this condition implies that r_p is the identity relation for p large enough. Now let $A(K)$ be the set of lattice points $A = (A_1, \ldots, A_s) \in \Lambda^s$ such that $A_i - A_j \in p\Lambda$ and $p \geq K|A| > 0$ for some i, j and $p \in \mathscr{Q}$. Then a lattice point $U \in \Lambda$ exists such that $A_i - A_j = pU$. Hence it follows that $p|U| = |A_i - A_j| \leq |A| \leq p/K$. However, applying this last inequality with $K = 2$ it follows that $U = 0$, i.e., $A_i = A_j$. The set $A(K)$ is then subdimensional, i.e., it is a subset of a finite number of hyperplanes, and consequently its density is null. To prove the existence part of the proposition notice that the product formula is positive if and only if $|[s]/r_p| \leq p^d$ for all prime $p \in Q$, and $|[s]/r_p| = s$, for p large enough. On the other hand, these conditions must be satisfied since, for any configuration S of s points, we have $|S/p| \leq |\Lambda/p|$ which is equal to p^d and $|S/p| = |S|$ for p large enough since the coordinates of the points of S are bounded. □

Corollary 2.3. *Balanced configurations exist.*

Proof. To introduce balanced configurations of size s we argue as follows. Suppose we have indexed the classes of Λ/p with integers between 1 and p^d. Therefore we can attach to each point A of Λ a finite sequence of integers, say $l(A)$, which represent the various classes of Λ/p at which A belongs as the prime number p ranges over the sequence $2, 3, 5, \ldots, p_r$ of primes less than $s^{1/d}$. Let i be the operator of pointwise incrementation, i.e.,

$$i(x_1, x_2, \ldots, x_r) = (x_1 + 1, x_2 + 1, \ldots, x_r + 1)$$

where the entry $x_i + 1$ is computed modulo p_i^d. Let 1 be the sequence $(1, 1, \ldots, 1)$. According to the Realizability Theorem, a sequence of lattice points $A_1, \ldots, A_s$ exists such that

(i) A_k, A_l are visible modulo p for each prime $p \geq s^{1/d}$ and
(ii) $l(A_k) = i^{k-1}(1)$.

Since the p_i^d are pairwise relatively prime, this configuration is clearly balanced. □

Remark. We use the Realizability Theorem in a slightly stronger form. According to Theorem 2.1, if a representative of $\mathscr{E}$ exists, then we can always find a representative belonging to some periodic set, assuming that this period is relatively prime to any element of $\mathscr{Q}$. For example, we can impose on the A_i to verify the conditions $A_i - B_i \in p\Lambda$ where p ranges over a finite set of primes disjoint from $\mathcal{Q}$ and where the B_i are given lattice points. A similar remark applies to Rumsey's theorem.

We end this section by introducing some more notation and some technical points which will be useful in the next section.

2.3. $\mathcal{Q}$-Visibility of a Configuration

Let $U_{\mathcal{Q}}(S)$ be the set of lattice points which are visible modulo $\mathcal{Q}$ from at least one point of S. From Rumsey's theorem and the finite additivity of the density function it is clear that the set $U_{\mathcal{Q}}(S)$ admits a density. This density is called the *$\mathcal{Q}$-visibility* of the configuration S, and is denoted by $u(\mathcal{Q}, S)$. According to the inclusion/exclusion principle,

$$u(\mathcal{Q}, S) = \sum_{E \subseteq S, E \neq \varnothing} (-1)^{|E|+1} \prod_{p \in \mathcal{P}} \left(1 - \frac{|E/p|}{p^d}\right). \tag{4}$$

Note that two configurations which are equivalent modulo $\mathcal{Q}$ have the same $\mathcal{Q}$-visibility. Now let $S_1, \ldots, S_r$ and T be $r + 1$ finite subsets of Λ. In our subsequent analysis we encounter the set $U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \backslash U_{\mathcal{Q}}(T)$ of lattice points which, for each $i \leq r$, can see modulo $\mathcal{Q}$ at least one point of each set S_i and cannot see modulo $\mathcal{Q}$ any of the points of T. This set admits a density, denoted by $u(\mathcal{Q}, S_1, S_2, \ldots, S_r; T)$. We relate this density to the following *difference operator.* For $A \subseteq \Lambda$ we define the operator Δ_A on the set of functions F from the power set of Λ to $\mathbb{R}$ as follows:

$$\Delta_A F(X) = F(A \cup X) - F(X). \tag{5}$$

From the additivity of the density function it can be easily verified that $\Delta_A D(X) = D(A \backslash X)$ and that $\Delta_B \Delta_A D = -\Delta_{A \cap B} D$. By repeated application of the above equality we get

$$u(\mathcal{Q}, S_1, \ldots, S_r; T) = (-1)^{r+1} \Delta_{U_{\mathcal{Q}}(S_1)} \cdots \Delta_{U_{\mathcal{Q}}(S_r)} u(\mathcal{Q}, T). \tag{6}$$

Proposition 2.1. *Assume that $\mathcal{Q}$ is infinite. If $r^{1/d}$ is less than the minimal prime of $\mathcal{Q}$, then the set $U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \backslash U_{\mathcal{Q}}(T)$ admits a nonnull density.*

Proof. Let A_i be a lattice point of S_i and for each lattice point B of T let q_B be a prime number of $\mathcal{Q}$ such that A_i and B are visible modulo q_B for all i. Then let $\mathcal{Q}_1 = \mathcal{Q} \backslash \{q_B | B \in T\}$, and let S be the set of A_i. Then the set, say V, of lattice points A such that

(i) $A - B \in q_B \Lambda$ and
(ii) A is visible modulo $\mathcal{Q}_1$ from each of the A_i,

is a subset of $U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \backslash U_{\mathcal{Q}}(T)$. However, according to Rumsey's theorem the density of V is nonnull since, by hypothesis, $|S/p| \le r < p^d$ for each prime $p \in \mathcal{Q}_1$. $\square$

3. Reduction to an Integer Optimization Problem

The difficulty of the optimization problem previously stated is due not only to the way we specify and manipulate the locations of the cameras (this problem is now solved by the Realizability Theorem), but also on the formulation of $u(S)$ as an alternating sum in identity (3). The key idea in overcoming the inherent complexity of optimizing $u(S)$ lies in an inductive formula for computing $u(S)$.

Theorem 3.1 (Reduction Theorem). *For any set $\mathcal{Q}$ of primes, any $p \in \mathcal{Q}$, and any configuration S,*

$$p^d u(\mathcal{Q}, S) = \sum_{c \in \Lambda/p} u(\mathcal{Q}\backslash\{p\}, S\backslash c).$$

Proof. Let c range over Λ/p. It is clear that a point of c is $\mathcal{Q}$-visible from a point of S if and only if it is $\mathcal{Q}\backslash\{p\}$-visible from a point of $S\backslash c$, i.e., $U_{\mathcal{Q}}(S) \cap c = U_{\mathcal{Q}\backslash\{p\}}(S\backslash c) \cap c$. According to Proposition 2.1 the set $U_{\mathcal{Q}}(S) \cap c$ admits a density given by $u(\mathcal{Q}\backslash\{p\}, S\backslash c)/p^d$. Using the (finite) additivity of the density we get the formula given in the theorem. $\square$

A first application of the previous theorem is the following.

Theorem 3.2 (Finiteness Theorem). *A necessary condition for the optimality of a configuration S is that*

$$|S/p| = \min\{|S|, p^d\}, \qquad \forall p \in \mathcal{P}.$$

In particular, the cameras of an optimal configuration must be pairwise visible modulo p for all primes $p \ge |S|^{1/d}$.

Observe that this theorem proves our conjecture for all square free integer n divisible by a prime number $p \ge |S|^{1/d}$.

Proof. The inequality $|S/p| \le \min\{|S|, p^d\}$ is always true even if S is not an optimal configuration. Now we assume that this inequality is strict for some $p \in \mathcal{P}$ and we construct a better configuration as follows. Let $c_1 \in \Lambda/p$ be such that $S \cap c_1$ has at least two elements and split $S \cap c_1$ into two nonempty parts S_1, S_2. Since a coset $c_2 \in \Lambda/p$ whose intersection with S is empty exists, the Realizability

Theorem asserts that a configuration S' in bijection with S exists such that

(1) S and S' are equivalent modulo $\mathscr{P}\backslash\{p\}$,
(2) $S' \cap c_1 = S'_1$,
(3) $S' \cap c_2 = S'_2$, and
(4) $S' \cap c = (S \cap c)'$

for all $c \neq c_1, c_2 \in \Lambda/p$ where E' stands for the image of E under the canonical bijection of S and S'. Now let $u'(\cdot)$ stand for $u(\mathscr{P}\backslash\{p\}, \cdot)$; since S and S' are equivalent modulo $\mathscr{P}\backslash\{p\}$ we have, for all $E \subseteq S$, the equality $u'(E) = u'(E')$; it follows, according to the Reduction Theorem, that

$$p^d(u(S') - u(S)) = u'(S\backslash S_1) + u'(S\backslash S_2) - u'(S\backslash(S_1 \cup S_2)) - u'(S).$$

However, the right member of this equation is $-\Delta_{U_{\mathscr{P}'}(S_1)}\Delta_{U_{\mathscr{P}'}(S_2)}u'(S)$ where $\mathscr{P}' = \mathscr{P}\backslash\{p\}$ and Δ_A is the difference operator (5). According to (6) we can write

$$p^d(u(S') - u(S)) = u'(S_1, S_2, S\backslash S_1 \cup S_2),$$

which is, according to Proposition 2.1, positive. The proof of the theorem is complete. □

An immediate consequence of the Finiteness Theorem is that a solution to the camera placement problem exists and that the number of solutions is finite modulo the relation of equivalence modulo $\mathscr{P}$.

To each configuration S, let $\{p_1, \ldots, p_r\}$ be the sequence of prime numbers p such that $|S/p| \neq |S|$ and let $m = p_1 \cdots p_r$ be their product. We associate to S the family of integers (a_c) defined by

$$a_c = |S \cap c_1 \cap c_2 \cap \cdots \cap c_r|, \tag{7}$$

where the index $c = (c_1, \ldots, c_r)$ ranges over the set $\mathscr{C} := \Lambda/p_1 \times \cdots \times \Lambda/p_r$. The integer a_c is the number of cameras in the coset c of Λ/m. Conversely, the Realizability Theorem shows that given a family of numbers $(a_c)_{c \in \mathscr{C}}$ a configuration S of $s = \sum_c a_c$ points exists such that $|S/p| = |S|$ for $p \neq p_i$ and to which the family (a_c) is associated by the procedure described above.

Equipped with this new way of specifying a configuration of cameras we now give a new expression for the function $u(S)$ to be maximized. We introduce the *reduced density function*, defined on the subsets E of S by

$$u'(E) := u(\mathscr{P}\backslash\{p_1, \ldots, p_r\}, E), \tag{8}$$

and the family of *reduced configurations* $\mathscr{B}_c \subseteq S$ defined by

$$\mathscr{B}_c = S\backslash \bigcup_{i=1}^{r} c_i. \tag{9}$$

Then by a repeated application of the Reduction Theorem we get that the visibility of the configuration S is the mean of the $\mathscr{P}\backslash\{p_1, \ldots, p_r\}$-density of the $p_1^d \cdots p_r^d$ reduced configurations, i.e.,

$$m^d u(S) = \sum_{c \in \mathscr{C}} u'(\mathscr{B}_c), \tag{10}$$

where $m^d = p_1^d \cdots p_r^d$. Before we give the properties of the reduced density function we recall that a real function $f(e)$ is called *absolutely monotone* if $(-1)^{n+1}\Delta^n f(e) > 0$ for all natural numbers $n \geq 1$, where Δ^n is the standard notation of the calculus of finite differences [5]:

$$\Delta^1 f(x) = f(x+1) - f(x), \qquad \Delta^{n+1} f = \Delta^1(\Delta^n f).$$

In particular, an absolutely monotone function f is strictly increasing and strictly concave, i.e., $f(e+1) - f(e)$ is strictly decreasing as a function of e.

Theorem 3.3 (Optimization Theorem). *Let S be a configuration of s cameras and let u' and $(\mathscr{B}_c)$ be the corresponding reduced density function and family of reduced configurations associated to S. Then for $E \subseteq S$ the function $u'(E)$ depends only on the size $|E|$ of the set E. Let $u'(e) = u'(E)$, where $e = |E|$ and let $b_c = |\mathscr{B}_c|$. Then we can prove the following properties*:

1. *$u'(e)$ is absolutely monotone.*
2. *$m^d u(S) = \sum_{c \in \mathscr{C}} u'(b_c)$.*
3. *$b_c = \sum_{h(c,c')=r} a_{c'}$ where the* Hamming distance $h(c, c')$ *is defined as the number of i such that $c_i \neq c'_i$.*
4. *$\sum_{c \in \mathscr{C}} b_c = s \prod_{i=1}^r (p_i^d - 1)$.*

Proof. By hypothesis, $|S/p| = |S|$ for all primes $p \notin \{p_1, \ldots, p_r\}$. This implies that for such primes p any two cameras in S are pairwise visible modulo p. In particular, for any set $E \subseteq S$, $|E/p| = |E|$. We conclude from (4) that the density function $u(\mathscr{P}\backslash\{p_1, \ldots, p_r\}, E)$ depends only on the cardinality of the set E. More precisely we have

$$u'(E) = \sum_{k=1}^{|E|} (-1)^{k+1} \binom{|E|}{k} \prod_{p \neq p_1, \ldots, p_r} \left(1 - \frac{k}{p^d}\right). \tag{11}$$

This proves the main assertion of the theorem regarding the function u'. Next we proceed to the second part of the theorem. In view of the previous observations the function $u'(e)$ represents the $\mathscr{P}\backslash\{p_1, \ldots, p_r\}$ density of a set of e cameras which are pairwise visible modulo p for each prime $p \in \mathscr{P}\backslash\{p_1, \ldots, p_r\}$. Let $E \subseteq S$ of cardinality e and let $A_1, \ldots, A_n$ be n points of $S\backslash E$. According to (6) and Proposition 2.1 we have

$$(-1)^{n+1}\Delta^n u'(e) = u'(\{A_1\}, \ldots, \{A_n\}; E) > 0.$$

Parts 2 and 3 are trivial reformulations of (7), (9), and (10). Finally, the last part follows from the following identities:

$$\begin{aligned}\sum_c b_c &= \sum_{c'} (a_{c'} |\{c \mid h(c, c') = r\}|) \\ &= |\{c \mid h(c, c') = r\}| \sum_{c'} a_{c'} \\ &= s \prod_{i=1}^{r} (p_i^d - 1).\end{aligned}$$

This completes the proof of the theorem. □

4. Optimal Configurations of Size up to 3^d

Now it is possible to give characterizations of optimal configurations of size at most 3^d.

Theorem 4.1. *A configuration of size at most 2^d is optimal if and only if its cameras are pairwise visible.*

Proof. Let S be an optimal configuration. From the Finiteness Theorem and our hypothesis $|S| \leq 2^d$, the cameras are pairwise visible modulo p for all prime numbers p. Consequently, the cameras are pairwise visible. Conversely, if the cameras are pairwise visible, then their visibility is uniquely determined. □

Theorem 4.2. *A configuration of size $\leq 3^d$ is optimal if and only if the configuration is balanced.*

Proof. Let S be an optimal configuration. From the Finiteness Theorem and the hypothesis, $|S| \leq 3^d$, the cameras are pairwise visible modulo each prime $p \geq 3$. It remains to determine the visibility modulo 2. According to the Optimization Theorem this is equivalent to solving the integer optimization problem:

$$\begin{aligned}&\text{maximize } u'(b_1) + u'(b_2) + \cdots + u'(b_{2^d}) \\ &\text{subject to } \begin{cases} b_i = \sum_{k \neq i} a_k, \\ a_1 + \cdots + a_{2^d} = s, \quad a_i \in \mathbb{N}. \end{cases}\end{aligned} \tag{12}$$

Here u' (the reduced density function) is an absolutely monotone function and a_i is interpreted as the number of cameras in the ith coset of $\Lambda/2$, provided we have numbered the 2^d cosets of $\Lambda/2$. In particular, u' is strictly concave, i.e., $\Delta u'(e) = u'(e + 1) - u'(e)$ is strictly decreasing. To show that the optimal solution

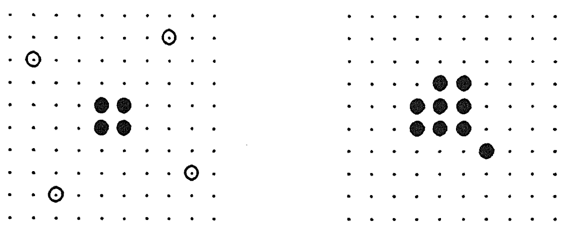

Fig. 3. Optimal configurations of four and nine cameras in the plane.

is obtained when $|a_i - a_j| \leq 1$ we proceed as follows. Assume that for some i, j we have $a_i > a_j + 1$ ($\Leftrightarrow b_i < b_j - 1$). Then we claim that the objective function is increased by replacing a_i and a_j by $a_i - 1$ and $a_j + 1$. Indeed, the variation of the objective function is $\Delta u'(b_i) - \Delta u'(b_j - 1)$ which is > 0 since u' is strictly concave. To conclude the proof it remains to observe that the condition $|a_i - a_j| \leq 1$ determines the value of the objective funcion. □

Figure 3 depict optimal configurations of four and nine cameras in dimension 2, respectively, while Fig. 2 depicts an optimal configuration of 27 cameras in dimension 3.

5. Conclusion

In this article we defined and analyzed the camera placement problem in complete integer lattices. We have reduced the combinatorial part of the problem to an instance of a general integer optimization problem involving absolute monotone functions; this enables us to characterize in simple terms the optimal configuration of size up to 3^d in d dimensions. By making a deeper and more elaborate analysis it is possible to provide a characterization of optimal configurations of size up to 5^d. Details of this proof can be found either in [13] or in more updated form in [10]. The case $s > 5^d$ remains open. For additional questions and considerations we refer the reader to [9], [13], and [10].

Acknowledgments

We are grateful to A. M. Odlyzko for suggesting reference [15] and to P. Flajolet and J. Berstel for useful conversations.

References

1. H. L. Abbott. Some results in combinatorial geometry. *Discrete Mathematics*, 9:199–204, 1974.
2. V. Boltjansky and I. Gohberg. *Results and Problems in Combinatorial Geometry*. Cambridge University Press, Cambridge, 1985.

3. P. Erdős, P. M. Gruber, and J. Hammer. *Lattice Points.* Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 39. Longman Scientific and Technical, Harlow, 1989.
4. J. Hammer. *Unsolved Problems Concerning Lattice Points.* Research Notes in Mathematics. Pitman, London, 1977.
5. C. Jordan. *Calculus of Finite Differences.* Chelsea, New York, 1965.
6. M. Kac. *Statistical Independence in Probability, Analysis and Number Theory.* The Carus Mathematical Monographs, Vol. 12. Mathematical Association of America, Washington, DC, 1959.
7. D. Knuth. *The Art of Computer Programming: Seminumerical Algorithms*, 2nd edn. Computer Science and Information Processing. Addison-Wesley, Reading, MA, 1981.
8. E. Kranakis and M. Pocchiola. Enumeration and visibility problems in integer lattices. *Proceedings of the 6th Annual ACM Symposium on Computational Geometry*, 1990, pp. 261–270.
9. E. Kranakis and M. Pocchiola. A brief survey of art gallery problems in integer lattice systems. *CWI Quarterly*, 4(4):269–282, 1991.
10. E. Kranakis and M. Pocchiola. Camera Placement in Integer Lattices. Technical Report 92-20, Lab. Inform., Ecole Normale Supérieure, Paris, 1992. Also available as TR 214, Carleton University, School of Computer Science.
11. W. O. J. Moser. Problems on extremal properties of a finite set of points. In *Discrete Geometry and Convexity*, Goodman *et al.*, eds. New York Academy of Sciences, Washington, DC, 1985, pp. 52–64.
12. J. O'Rourke. *Art Gallery Theorems and Algorithms.* International Series of Monographs on Computer Science. Oxford University press, Oxford, 1987.
13. M. Pocchiola. Trois thèmes sur la visibilité: énumération, optimisation et graphique 2D. Technical Report 90-23, Lab. Inform., Ecole Normale Supérieure, Paris, 1990. Ph.D. thesis.
14. D. F. Rearick. Mutually visible lattice points. *Norske Vid. Selsk. Forh.* (*Trondheim*), 39:41–45, 1966.
15. H. Rumsey, Jr. Sets of visible points. *Duke Mathematical Journal*, 33:263–274, 1966.

Received October 9, 1992, and in revised form March 12, 1993, and January 1, 1994.